CIRCLE IMMERSIONS THAT CAN BE DIVIDED INTO TWO ARC EMBEDDINGS

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Abstract. We give a complete characterization of a circle immersion that can be divided into two arc embeddings in terms of its chord diagram.

1. Introduction

Let $\mathbb{S}^1$ be the unit circle. Let X be a set and $f : \mathbb{S}^1 \to X$ a map. Let n be a natural number greater than one. Suppose that there are n subspaces $I_1, \cdots, I_n$ of $\mathbb{S}^1$ with the following properties.

(1) Each I_i is homeomorphic to a closed interval.

(2) $\mathbb{S}^1 = I_1 \cup \cdots \cup I_n$.

(3) The restriction map $f|_{I_i} : I_i \to X$ is injective for each i.

Then we say that f can be divided into n arc embeddings. We define the *arc number* of f, denoted by $\mathrm{arc}(f)$, to be the smallest such n except the case that f itself is injective. If f itself is injective then we define $\mathrm{arc}(f) = 1$. If f cannot be divided into n arc embeddings for any natural number n then we define $\mathrm{arc}(f) = \infty$. Note that if f can be divided into n arc embeddings then there exist n subspaces $I_1, \cdots, I_n$ of $\mathbb{S}^1$ with (1), (2) and (3) above together with the following additional condition.

(4) $I_i \cap I_j = \partial I_i \cap \partial I_j$ for each i and j with $1 \leq i < j \leq n$.

Namely we may assume that $\mathbb{S}^1$ is covered by mutually interior disjoint n simple arcs $I_1, \cdots, I_n$.

Let $\mathrm{S}(f) = \{x \in \mathbb{S}^1 | f^{-1}(f(x)) \text{ is not a singleton.}\}$ and $\mathrm{s}(f) = f(\mathrm{S}(f))$. We say that a map $f : \mathbb{S}^1 \to X$ has *finite multiplicity* if $\mathrm{S}(f)$ is a finite subset of $\mathbb{S}^1$. From now on we restrict our attention to maps that have finite multiplicity. The purpose of this paper is to give a characterization of a map $f : \mathbb{S}^1 \to X$ with $\mathrm{arc}(f) = 2$. By $|Y|$ we denote the cardinality of a set Y. Let $m(f)$ be the maximum of $|f^{-1}(y)|$ where y varies over all points of X. It is clear that $\mathrm{arc}(f) \geq m(f)$. Thus we further restrict our attention to a map $f : \mathbb{S}^1 \to X$ whose multiple points are only finitely many double points. Namely f has finite multiplicity and $m(f) \leq 2$. Then we have $|\mathrm{S}(f)| = 2m$ for some non-negative integer m. Then the *crossing number* of f, denoted by $c(f)$, is defined by m.

Let m be a natural number. An *m-chord diagram* on $\mathbb{S}^1$ is a pair $\mathcal{C} = (P, \varphi)$ where P is a subset of $\mathbb{S}^1$ that contains exactly $2m$ points and φ is a fixed point free involution on P. A *chord* c of $\mathcal{C}$ is an unordered pair of points $(x, \varphi(x)) = (\varphi(x), x)$ where x is a point in P. Let $\sim_{\mathcal{C}}$ be the equivalence relation on $\mathbb{S}^1$ generated by

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$x \sim_{\mathcal{C}} \varphi(x)$ for every $x \in P$. Let $\mathbb{S}^1/\sim_{\mathcal{C}}$ be the quotient space and $f_{\mathcal{C}} : \mathbb{S}^1 \to \mathbb{S}^1/\sim_{\mathcal{C}}$ the quotient map. We call $f_{\mathcal{C}}$ the associated map of $\mathcal{C}$. Then the arc number of $\mathcal{C}$, denoted by $\mathrm{arc}(\mathcal{C})$, is defined to be the arc number of $f_{\mathcal{C}}$. Two m-chord diagrams $\mathcal{C}_1 = (P_1, \varphi_1)$ and $\mathcal{C}_2 = (P_2, \varphi_2)$ are *equivalent* if there is an orientation preserving self-homeomorphism h of $\mathbb{S}^1$ such that $h(P_1) = P_2$ and $h \circ \varphi_1 = \varphi_2 \circ h$. From now on we consider m-chord diagrams up to this equivalence relation. In the following we sometimes express an m-chord diagram $\mathcal{C} = (P, \varphi)$ by m line-segments in the plane $\mathbb{R}^2$ where $\mathbb{S}^1 \subset \mathbb{R}^2$, P is the set of the end points of these line-segments and x and $\varphi(x)$ are joined by a line segment for each $x \in P$. Thus a line segment express a chord and from now on we do not distinguish them. See for example Figure 1.1.

Let $f : \mathbb{S}^1 \to X$ be a map whose multiple points are only finitely many double points. By $\mathcal{C}(f)$ we denote the $c(f)$-chord diagram $(\mathrm{S}(f), \varphi_f)$ where $\varphi_f : \mathrm{S}(f) \to \mathrm{S}(f)$ is the fixed point free involution with $f|_{\mathrm{S}(f)} \circ \varphi_f = f|_{\mathrm{S}(f)}$. We call $\mathcal{C}(f)$ the associated chord diagram of f. Then it is clear that $\mathrm{arc}(f) = \mathrm{arc}(\mathcal{C}(f))$.

A chord diagram $\mathcal{D} = (Q, \psi)$ is called a *sub-chord diagram* of a chord diagram $\mathcal{C} = (P, \varphi)$ if Q is a subset of P and ψ is the restriction of φ on Q. Then it is clear that $\mathrm{arc}(\mathcal{D}) \leq \mathrm{arc}(\mathcal{C})$. We call $\mathcal{D} = (Q, \psi)$ a *proper sub-chord diagram* of $\mathcal{C} = (P, \varphi)$ if $\mathcal{D}$ is a sub-chord diagram of $\mathcal{C}$ and Q is a proper subset of P.

Let n be a natural number. Let $\mathcal{C}_{2n+1}$ be a $(2n+1)$-chord diagram as illustrated in Figure 1.1. To give a more precise definition we introduce the followings. Let k be a natural number greater than two. Let R_k be a regular k-gon inscribed in $\mathbb{S}^1$ and $v_{k;1}, \cdots, v_{k;k}$ the vertices of R_k that are arranged in this order on $\mathbb{S}^1$ along the counterclockwise orientation of $\mathbb{S}^1$. Namely $v_{k;i}$ and $v_{k;i+1}$ are adjacent in R_k for each i where the indices are considered modulo k. Let j is a natural number less than $\frac{k}{2}$. Let $c(k; i, j)$ be the chord joining $v_{k;i}$ and $v_{k;i+j}$ for each $i \in \{1, \cdots, k\}$. Then $\mathcal{C}_{2n+1}$ is the chord diagram represented by chords $c(4n+2; 2i-1, 2n-1)$ with $i \in \{1, \cdots, 2n+1\}$. We will show that $\mathrm{arc}(\mathcal{C}_{2n+1}) = 3$ but $\mathrm{arc}(\mathcal{D}) = 2$ for any proper sub-chord diagram $\mathcal{D}$ of $\mathcal{C}_{2n+1}$. Then we have the following theorem.

Theorem 1.1. *Let m be a natural number and $\mathcal{C}$ an m-chord diagram on $\mathbb{S}^1$. Then $\mathrm{arc}(\mathcal{C}) = 2$ if and only if no sub-chord diagram of $\mathcal{C}$ is equivalent to the chord diagram $\mathcal{C}_{2n+1}$ for any natural number n.*

The motive for this paper was the result in [2] that every knot has a diagram which can be divided into two simple arcs. This result is re-discovered by [3] and [4]. See also [1]. Then it is natural to ask what plane closed curve can be divided into two simple arcs. Theorem 1.1 gives an answer to this question. However we still have a question whether or not do we actually need all of $\mathcal{C}_3, \mathcal{C}_5, \cdots$. The following proposition answers this question that we actually need all of them.

Proposition 1.2. *For each natural number n there exist a smooth immersion $f_n : \mathbb{S}^1 \to \mathbb{R}^2$ with $\mathrm{arc}(f_n) = 3$ that has only finitely many transversal double points such that the associated chord diagram $\mathcal{C}(f_n)$ of f_n has a sub-chord diagram which is equivalent to $\mathcal{C}_{2n+1}$ but has no sub-chord diagram which is equivalent to $\mathcal{C}_{2m+1}$ for any $m < n$.*

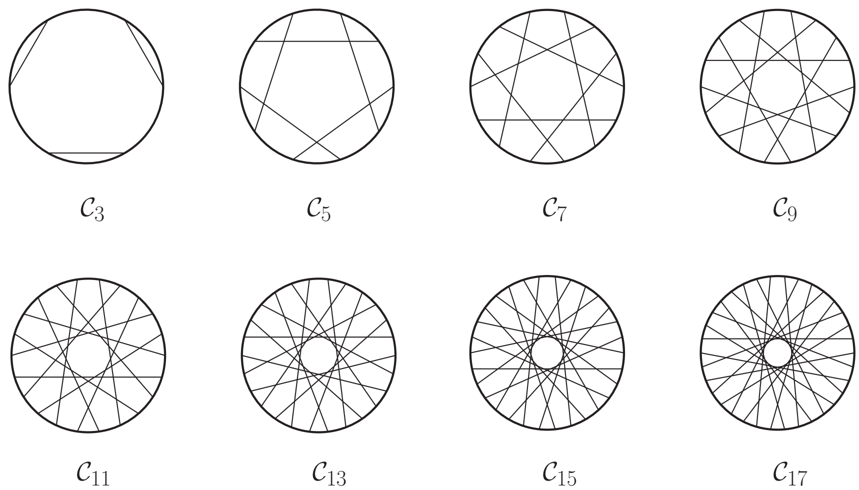

FIGURE 1.1.

2. Proof of Theorem 1.1

First we check that $\mathrm{arc}(\mathcal{C}_{2n+1}) = 3$. Let $\mathcal{C} = (P, \varphi)$ be an m-chord diagram with $\mathrm{arc}(\mathcal{C}) = 2$. A pair of points $p, q \in \mathbb{S}^1 \setminus P$ are called a *cutting pair* of $\mathcal{C}$ if x and $\varphi(x)$ belong to the different components of $\mathbb{S}^1 \setminus \{p, q\}$ for each $x \in P$. Then we have that p and q are "antipodal". Namely we have that each component of $\mathbb{S}^1 \setminus \{p, q\}$ contains exactly m points of P. Thus we can check whether or not a given m-chord diagram has arc number 2 by examining m pairs of antipodal points of it. Then by the symmetry of $\mathcal{C}_{2n+1}$ we immediately have that $\mathrm{arc}(\mathcal{C}_{2n+1}) > 2$. Then it is easily seen that $\mathrm{arc}(\mathcal{C}_{2n+1}) = 3$ and $\mathrm{arc}(\mathcal{D}) = 2$ for any proper sub-chord diagram $\mathcal{D}$ of $\mathcal{C}_{2n+1}$. Then the 'only if part' of the proof of Theorem 1.1 immediately follows. The 'if part' immediately follows from the following proposition.

Proposition 2.1. *Let $\mathcal{C}$ be a chord diagram on $\mathbb{S}^1$ that satisfies the following condition* $(*)$.

$(*)$ $\mathrm{arc}(\mathcal{C}) \geq 3$ *and* $\mathrm{arc}(\mathcal{D}) = 2$ *for any proper sub-chord diagram $\mathcal{D}$ of $\mathcal{C}$.*

Then there is a natural number n such that $\mathcal{C}$ is equivalent to $\mathcal{C}_{2n+1}$.

Note that deleting a chord will decrease the arc number at most by one. Therefore, if $\mathcal{C}$ is a chord diagram on $\mathbb{S}^1$ that satisfies the condition $(*)$, then $\mathrm{arc}(\mathcal{C}) = 3$. For the proof of Proposition 2.1 we prepare the following lemmas. Let $\mathcal{C} = (P, \varphi)$ be a chord diagram and $c = (x, \varphi(x))$ a chord of $\mathcal{C}$. Let α and β be the components of $\mathbb{S}^1 \setminus \{x, \varphi(x)\}$. We may suppose without loss of generality that $|\alpha \cap P| \leq |\beta \cap P|$. Then the *length* of c in $\mathcal{C}$, denoted by $l(c) = l(c, \mathcal{C})$, is defined to be $|\alpha \cap P| + 1$. By $\mathcal{C} \setminus c$, we denote the chord diagram $(P \setminus \{x, \varphi(x)\}, \varphi|_{P \setminus \{x, \varphi(x)\}})$. Let p, q, x and y be mutually distinct four points on $\mathbb{S}^1$. We say that the pair of points p and q separates the pair of points x and y if each component of $\mathbb{S}^1 \setminus \{p, q\}$ contains exactly one of x and y. Note that the pair of points p and q separates the pair of points x and y if and only if the chord joining p and q intersects the chord joining x and y.

Lemma 2.2. *Let $\mathcal{C} = (P, \varphi)$ be a chord diagram on $\mathbb{S}^1$ that satisfies the condition $(*)$. Let $c = (x, \varphi(x))$ be a chord of $\mathcal{C}$. Let p and q be a cutting pair of $\mathcal{C} \setminus c$. Then p and q do not separate x and $\varphi(x)$.*

Proof. If p and q separate x and $\varphi(x)$, then p and q is a cutting pair of $\mathcal{C}$ itself. Then it follows $\mathrm{arc}(\mathcal{C}) = 2$. This is a contradiction. $\square$

Lemma 2.3. *Let $\mathcal{C} = (P, \varphi)$ be an m-chord diagram on $\mathbb{S}^1$ that satisfies the condition $(*)$. Let $c = (x, \varphi(x))$ be a chord of $\mathcal{C}$. Then $l(c, \mathcal{C}) \leq m - 2$.*

Proof. Since $|P| = 2m$ we have $1 \leq l(c, \mathcal{C}) \leq m$ for any chord c. First we examine the case $l(c, \mathcal{C}) = m$. In this case we have that each component of $\mathbb{S}^1 \setminus \{x, \varphi(x)\}$ contains exactly $m - 1$ elements of P. Let p and q be a cutting pair of $\mathcal{C} \setminus c$. Then by Lemma 2.2 we have that p and q do not separate x and $\varphi(x)$. Note that each component of $\mathbb{S}^1 \setminus \{p, q\}$ also contains exactly $m - 1$ elements of P. Then it follows that p and q are next to x and $\varphi(x)$ or $\varphi(x)$ and x respectively. We may suppose without loss of generality that p and q are next to x and $\varphi(x)$ respectively. Let p' be a point on $\mathbb{S}^1$ that is next to x and such that p' and p separate x and $\varphi(x)$. Then we have that p' and q is a cutting pair of $\mathcal{C}$. This is a contradiction. Next we examine the case $l(c, \mathcal{C}) = m - 1$. In this case we have that one component of $\mathbb{S}^1 \setminus \{x, \varphi(x)\}$ contains exactly $m - 2$ elements of P and the other component contains exactly m elements of P. Let p and q be a cutting pair of $\mathcal{C} \setminus c$. Then by Lemma 2.2 we have that p and q do not separate x and $\varphi(x)$. Note that each component of $\mathbb{S}^1 \setminus \{p, q\}$ contains exactly $m - 1$ elements of P. Then it follows that one of p and q, say p is next to x or $\varphi(x)$, say x. Let p' be a point on $\mathbb{S}^1$ that is next to x and such that p' and p separate x and $\varphi(x)$. Then we have that p' and q is a cutting pair of $\mathcal{C}$. This is a contradiction. Thus we have $l(c, \mathcal{C}) \leq m - 2$. $\square$

Lemma 2.4. *Let $\mathcal{C} = (P, \varphi)$ be an m-chord diagram on $\mathbb{S}^1$ that satisfies the condition $(*)$. Let $c = (x, \varphi(x))$ be a chord of $\mathcal{C}$. Then $l(c, \mathcal{C}) \geq m - 2$.*

Proof. Suppose that there is a chord $c = (x, \varphi(x))$ of $\mathcal{C}$ with $l(c, \mathcal{C}) \leq m - 3$. Let $\mathcal{D} = (Q, \varphi|_Q)$ be the maximal sub-chord diagram of $\mathcal{C}$ such that $x, \varphi(x) \in Q$ and $l(c, \mathcal{D}) = 1$. Let n be the number of chords of $\mathcal{D}$. Let A (resp. B) be the point in Q such that each components of $\mathbb{S}^1 \setminus \{x, A\}$ (resp. $\mathbb{S}^1 \setminus \{\varphi(x), B\}$) contains $n-1$ points of Q. Note that $n \geq m - (l(c, \mathcal{C}) - 1) \geq m - (m - 3 - 1) = 4$. Therefore we have that $\mathcal{D}$ has at least 4 chords. Then there is a chord $d = (y, \varphi(y))$ of $\mathcal{D}$ such that $\{y, \varphi(y)\}$ and $\{x, \varphi(x), A, B\}$ are mutually disjoint. Suppose that y and $\varphi(y)$ do not separate x and A. In this case there must be a chord $e = (z, \varphi(z))$ of $\mathcal{D}$ such that $\{z, \varphi(z)\}$ and $\{x, \varphi(x), y, \varphi(y)\}$ are mutually disjoint and z and $\varphi(z)$ do not separate y and $\varphi(y)$. Then we have that the chords c, d and e form a sub-chord diagram of $\mathcal{C}$ that is equivalent to $\mathcal{C}_3$. This is a contradiction. Suppose that y and $\varphi(y)$ separate x and A. Let p and q be a cutting pair of $\mathcal{C} \setminus d$. Then we have by Lemma 2.2 that p and q do not separate y and $\varphi(y)$. Note that p and q is also a cutting pair of $\mathcal{D} \setminus d$ and they separate x and $\varphi(x)$. Then we have that the component of $\mathbb{S}^1 \setminus \{p, q\}$ that contains both A and B has more points of $Q \setminus \{y, \varphi(y)\}$ than the other. This is a contradiction. $\square$

Thus we have shown the following lemma.

Lemma 2.5. *Let $\mathcal{C} = (P, \varphi)$ be an m-chord diagram on $\mathbb{S}^1$ that satisfies the condition $(*)$. Then $\mathcal{C}$ satisfies the following condition $(\star)$.*

$(\star)$ *$l(c, \mathcal{C}) = m - 2$ for every chord c of $\mathcal{C}$.*

Proposition 2.6. *Let $\mathcal{C} = (P, \varphi)$ be an m-chord diagram on $\mathbb{S}^1$ that satisfies the condition $(\star)$. If m is even then m is divisible by 4 and $\mathrm{arc}(\mathcal{C}) = 2$. If m is odd then $\mathcal{C}$ is equivalent to $\mathcal{C}_m$.*

Proof. Recall that R_{2m} is a regular $(2m)$-gon inscribed in $\mathbb{S}^1$ and $v_{2m;1}, \cdots, v_{2m;2m}$ are the vertices of R_{2m} lying in this order. Let $G_{2m,m-2}$ be the graph whose vertices are $v_{2m;1}, \cdots, v_{2m;2m}$ and whose edges are the chords $c(2m; i, m-2)$ joining the vertices $v_{2m;i}$ and $v_{2m;i+m-2}$ where $i \in \{1, \cdots, 2m\}$. By calculating the greatest common divisor $(2m, m-2) = (2m - 2(m-2), m-2) = (4, m-2)$ we have the isomorphism type of the graph $G_{2m,m-2}$ as follows.

(1) If m is a multiple of 4 then $(4, m-2) = 2$ and therefore $G_{2m,m-2}$ is isomorphic to a disjoint union of two m-cycles.

(2) If m is congruent to 2 modulo 4 then $(4, m-2) = 4$ and therefore $G_{2m,m-2}$ is isomorphic to a disjoint union of four $\frac{m}{2}$-cycles.

(3) If m is odd then then $(4, m-2) = 1$ and therefore $G_{2m,m-2}$ is isomorphic to a $2m$-cycle.

Note that in each case $\mathcal{C}$ must be a complete matching of the graph $G_{2m,m-2}$. In (1) we have up to symmetry that $\mathcal{C}$ is as illustrated in Figure 2.1. Then we have that $\mathrm{arc}(\mathcal{C}) = 2$. In (2) we have that $G_{2m,m-2}$ has no complete matchings because an $\frac{m}{2}$-cycle is an odd-cycle. In (3) we have that $\mathcal{C}$ is equivalent to $\mathcal{C}_m$. This completes the proof. □

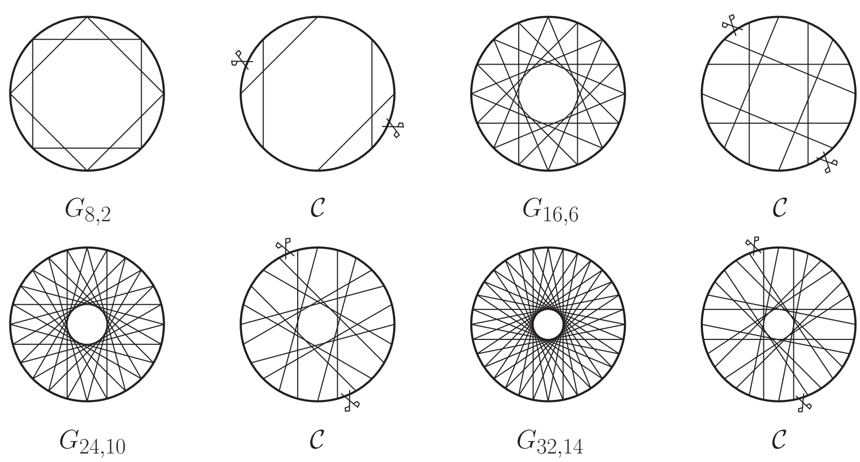

FIGURE 2.1.

Proof of Proposition 2.1. By Lemma 2.5 and Proposition 2.6 we have the result. □

3. Examples of plane curves

Let $\mathcal{C} = (P, \varphi)$ be a chord diagram and $c = (x, \varphi(x))$ and $d = (y, \varphi(y))$ two chords of $\mathcal{C}$. We say that c and d are *parallel* if the pair of points x and $\varphi(x)$ do not separate the pair of points y and $\varphi(y)$. We say that two distinct points x and y in P are *next to each other* if there is a component of $\mathbb{S}^1 \setminus \{x, y\}$ that is disjoint from P. We say that c and d are *close to each other* if x and y are next to each other and $\varphi(x)$ and $\varphi(y)$ are next to each other, or x and $\varphi(y)$ are next to each other and $\varphi(x)$ and y are next to each other.

Proof of Proposition 1.2. The cases $n = 1, 2$ are shown in Figure 3.1. We consider the case $n \geq 3$. Let $G_{2n+1} = \mathbb{S}^1 / \sim_{\mathcal{C}_{2n+1}}$ be the 4-regular graph obtained from $\mathbb{S}^1$ by identifying the end points of each chord of $\mathcal{C}_{2n+1}$. It is easy to observe that G_{2n+1} is isomorphic to a graph obtained from a $(2n+1)$-cycle Γ_{2n+1} on vertices $v_1, \cdots, v_{2n+1}$ lying in this order by adding edges joining v_i and v_{i+3} for each i such that along the counterclockwise orientation of $\mathbb{S}^1$ the vertices of G_{2n+1} appears $v_i, v_{i+1}, v_{i+1-3}, v_{i+1-3+1}, v_{i+1-3+1-3}, \cdots$. See Figure 3.2. Then we deform them on $\mathbb{R}^2$ as illustrated in Figure 3.3. Note that they are classified into three types by $2n+1$ modulo 6. Namely G_{2n+1+6} is obtained from G_{2n+1} by cutting open G_{2n+1} along the dotted line and inserting two pieces of a pattern as illustrated in Figure 3.3. We modify this G_{2n+1} and have the image $f_n(\mathbb{S}^1)$ as illustrated in Figure 3.4. Note that each vertex of G_{2n+1} is replaced by two transversal double points. We call them a *twin pair*. The chords corresponding to them are also called a twin pair. By choosing any one of them for each twin pair we have a sub-chord diagram of $\mathcal{C}(f_n)$ that is equivalent to $\mathcal{C}_{2n+1}$ by the construction. Observe that each $f_n(\mathbb{S}^1)$ is made of $(2n+1)$-times repetitions of "one step forward and three steps back" along the $(2n+1)$-cycle Γ_{2n+1} and it totally goes around Γ_{2n+1} twice. Here "one step forward" corresponds to an edge of G_{2n+1} joining v_i and v_{i+1} and "three steps back" corresponds to an edge of G_{2n+1} joining v_{i+1} and v_{i+1-3}. It has no local double points and each double point comes from a part and another part that is one lap behind. Therefore we have that $\mathrm{arc}(f_n) = 3$.

Now we will check that no sub-chord diagram of $\mathcal{C}(f_n)$ is equivalent to $\mathcal{C}_{2m+1}$ for any $m < n$. Note that two chords in a twin pair are close to each other in $\mathcal{C}(f_n)$. Let $\mathcal{D}(f_n)$ be a sub-chord diagram of $\mathcal{C}(f_n)$ obtained from $\mathcal{C}(f_n)$ by deleting one of two chords for each twin pair in $\mathcal{C}(f_n)$. Since no two chords in $\mathcal{C}_{2m+1}$ are close to each other it is sufficient to check that no sub-chord diagram of $\mathcal{D}(f_n)$ is equivalent to $\mathcal{C}_{2m+1}$ for any $m < n$. Suppose that $\mathcal{E}$ is a sub-chord diagram of $\mathcal{D}(f_n)$ that is equivalent to $\mathcal{C}_{2m+1}$ for some $m < n$. Since no proper sub-chord diagram of $\mathcal{C}_{2n+1}$ is equivalent to $\mathcal{C}_{2m+1}$ we have that there is a chord c of $\mathcal{E}$ that does not belong to any twin pair of $\mathcal{C}(f_n)$. Namely c corresponds to a transversal double point of f_n that comes from a double point of $G_{2n+1} \subset \mathbb{R}^2$ in Figure 3.3. Observe that for each chord $d = (x, \varphi(x))$ of $\mathcal{C}_{2m+1}$ there exist exactly two chords $g = (y, \varphi(y)$ and $h = (z, \varphi(z))$ of $\mathcal{C}_{2m+1}$ that are parallel to d such that all of $y, \varphi(y), z$ and $\varphi(z)$ are contained in the same component of $\mathbb{S}^1 \setminus \{x, \varphi(x)\}$. Therefore c must have such two chords in $\mathcal{D}(f_n)$. By the "one step forward and three steps back" structure of $f_n(\mathbb{S}^1)$ mentioned above the double points corresponding to such chords must lie in a small neighbourhood of the double point corresponding to c. Then we can check that there are no such two chords for c in $\mathcal{D}(f_n)$ except the case $2n+1$ is congruent to 5 modulo 6 and c is one of the three chords of $\mathcal{C}(f_n)$ that come from the three

double points on the same edge of $G_{2n+1} \subset \mathbb{R}^2$ in Figure 3.3. For this exceptional case we further observe that the chords g and h above intersect unless $m = 1$ and the end point of g (resp. h) that is next to x or $\varphi(x)$ in $\mathcal{C}_{2m+1}$ is not next to any end point of h (resp. g) in $\mathcal{C}_{2m+1}$. However in this exceptional case we can check that there are no such two chords for c in $\mathcal{D}(f_n)$. This is a contradiction. □

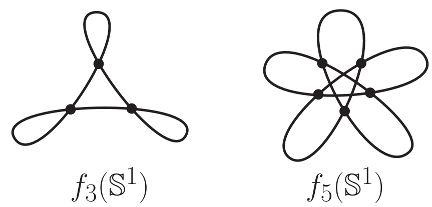

FIGURE 3.1.

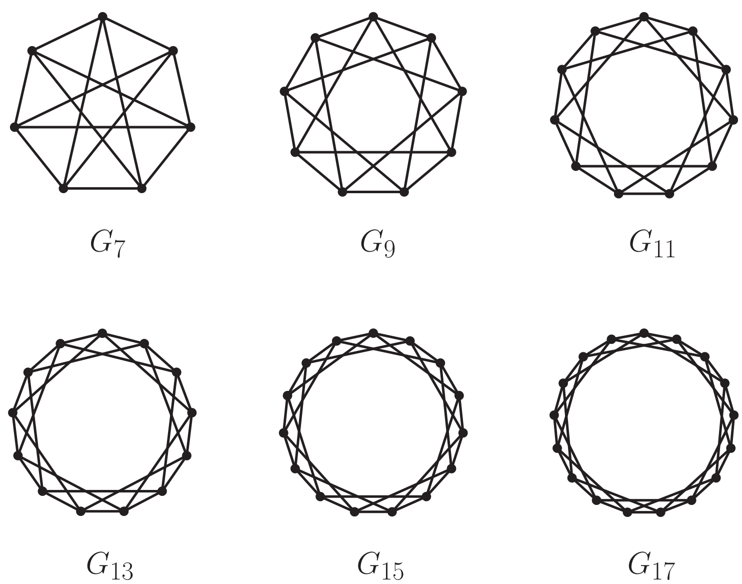

FIGURE 3.2.

Remark 3.1. It is easy to see that the graph $G_{2n+1} = \mathbb{S}^1/\sim_{\mathcal{C}_{2n+1}}$ is a non-planar graph for $n \geq 2$. Therefore we have that for $n \geq 2$ there is no smooth immersion $f : \mathbb{S}^1 \to \mathbb{R}^2$ that has only finitely many transversal double points whose associated chord diagram $\mathcal{C}(f)$ itself is equivalent to $\mathcal{C}_{2n+1}$.

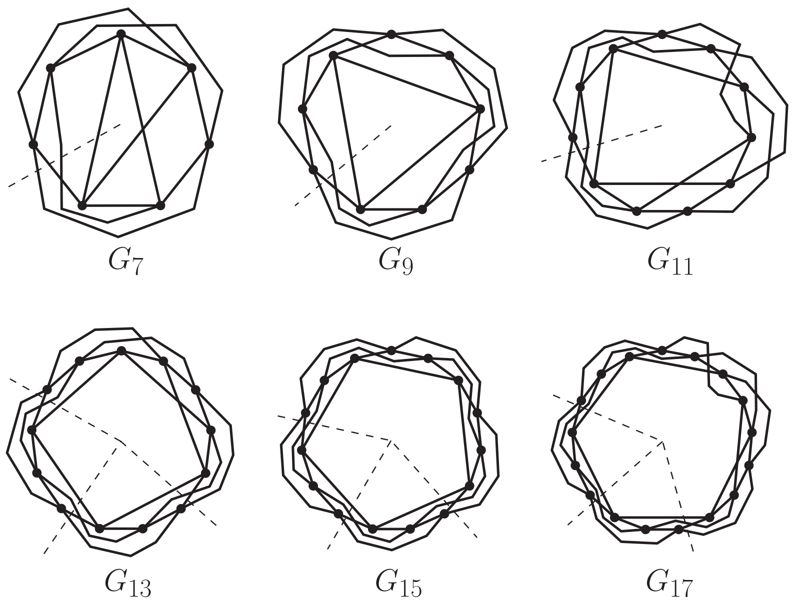

Figure 3.3.

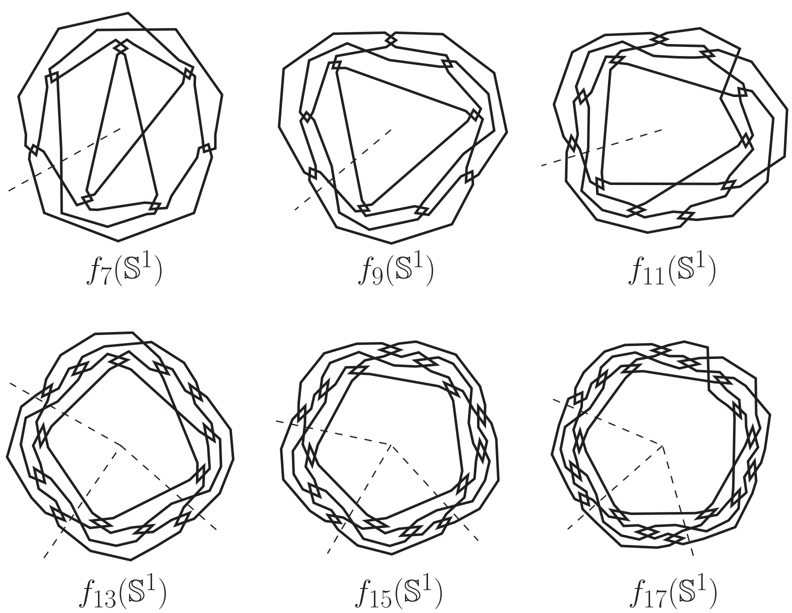

Figure 3.4.

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References

[1] C. Adams, R. Shinjo and K. Tanaka, Complementary regions of knot and link diagrams, arXiv:0812.2558 (2008).

[2] G. Hotz, Arkadenfadendarstellung von Knoten und eine neue Darstellung der Knotengruppe (German), *Abh. Math. Sem. Univ. Hamburg*, **24** (1960), 132-148.

[3] M. Ozawa, Edge number of knots and links, arXiv:0705.4348 (2007).

[4] R. Shinjo, Complementary regions of projections of spatial graphs, in preparation.

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